Apr 18

Barrier function

Let $(M, g(t))$, $t \in [0, T)$ with unif. bounded curvature

Lemma 7.44 (existence of barrier function)

barrier function

For $a>0$, $A>0$ Lemma 7.44 constructs a ϕ s.t.

① $$\left(\frac{\partial}{\partial t} - \Delta\right) \phi \geq A \phi \quad \text{on } M \times [0, T)$$

and $$e^{a \cdot d_{g(t)}(0, x)} \leq \phi(x) \leq e^{b \cdot (d_{g(t)}(0, x) + 1)}$$

for some $b>0$

with control on the derivatives of $\log \phi$

$$|\nabla^{g(t)} \log \phi|_{g(t)} \leq C,$$

$$|\nabla^{g(t)} \nabla^{g(t)} \log \phi|_{g(t)} \leq C \qquad \text{for some } C < \infty$$

Next: barrier function ϕ +

subsolution φ with at most exponential growth

$\longrightarrow$ max principle.

Apply the barrier function we have

Cor 7.45.

- $(M^n, g(t))$ $t \in [0,T]$
  complete sol. of RF with bdd Rm $|Rm(g(t))| < \infty$.
- φ smooth satisfies IC : $\varphi(x,0) \leq 0$
- whenever $\varphi(x,t) > 0$, φ is a subsolution

② $$\frac{\partial}{\partial t}\varphi \leq \Delta\varphi + C\varphi,$$

- growth control
  $p \in M$
  $$\exists\ \varphi(x,t) \leq e^{A_1(d_{g(t)}(x,p)+1)} \quad \forall\ (x,t)$$

$\triangleright$ $\varphi(x,t) \leq 0$ for all $x \in M$, $t \in [0,T]$.

pf. This is a special case of Cor 7.42,

A generalization of Thm 7.39

Corollary 7.43. *Let $(M^n, g(t))$, $t \in [0,T]$, be a complete solution of the Ricci flow with uniformly bounded curvature. If u is a weak subsolution of the heat equation on $M^n \times [0,T]$ with $u(\cdot,0) \leq 0$ and if*

$$\int_0^T \int_{M^n} \exp\left(-\alpha d^2_{g(0)}(x,O)\right) u^2_+(x,t)\, d\mu_{g(t)}(x)\, dt < \infty$$

for some $\alpha > 0$, then $u \leq 0$ on $M^n \times [0,T]$.

③

- Use Lemma 7.44 with $a > A_1$, $A > C$, $p = 0$ to construct ϕ s.t.

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right)\phi \geq A\phi \\ e^{a \cdot d_{g(t)}(x,p)} \leq \phi(x,t) \leq e^{b\cdot(d_{g(t)}(x,p)+1)}, \text{ for some } b > 0 \end{cases}$$

- if $\frac{\varphi}{\phi}(x,t) \geq 0$, then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\varphi}{\phi}\right)$$

$$= \underbrace{\frac{\left(\frac{\partial}{\partial t} - \Delta\right)\varphi}{\phi}}_{\leq \frac{C\varphi}{\phi} \text{ by ②}} \underbrace{- \frac{\varphi}{\phi^2}\underbrace{\left(\frac{\partial}{\partial t} - \Delta\right)\phi}_{\geq A\phi,\ A>0 \text{ by ①}}}_{\leq -\frac{\varphi}{\phi}\cdot A} + 2\nabla\log\phi \cdot \nabla\left(\frac{\varphi}{\phi}\right)$$

$$\leq (C-A)\frac{\varphi}{\phi} + 2\nabla\log\phi\cdot\nabla\left(\frac{\varphi}{\phi}\right)$$

by ③

$$(\star) \quad \leq 2\nabla\log\phi\cdot\nabla\left(\frac{\varphi}{\phi}\right)$$

- Use the proof of maximum principle on closed M

$$u_\varepsilon := \frac{\varphi}{\phi} - \varepsilon(1+t) \qquad \forall \varepsilon > 0$$

$u_\varepsilon(x,0) < 0$ as $\varphi(x,0) \leq 0$

mimic the proof of Prop 2.9 on p.100.

Claim 1 : $u_\varepsilon(x,t) < 0$ for small t.

note that

$$\frac{\varphi}{\phi}(x,t) \leq \exp\left\{A_1 + (A_1 - a)\, d_{g(t)}(x,p)\right\}.$$

Since $A_1 - a < 0$ and $d_{g(t)}(x,p) \geq c d_{g(0)}(x,p)$ for all $x \in M$ and $t \in [0,T]$ (since Rc is uniformly bounded), we have

$$\frac{\varphi}{\phi}(x,t) \leq e^{-A_1}$$

for all $x \in M$ with $d_{g(0)}(x,p) \geq \frac{2A_1}{c(a-A_1)}$ and $t \in [0,T]$. Thus

(7.35) $$u_\varepsilon < 0 \text{ in } \left(M - B_{g(0)}\left(p, \frac{2A_1}{c(a-A_1)}\right)\right) \times [0,T].$$

On the other hand, since u_ε is continuous and $\overline{B_{g(0)}\left(p, \frac{2A_1}{c(a-A_1)}\right)}$ is compact, for $\tau > 0$ sufficiently small, we have

(7.36) $$u_\varepsilon < 0 \text{ in } \overline{B_{g(0)}\left(p, \frac{2A_1}{c(a-A_1)}\right)} \times [0,\tau].$$

This proves the claim.

(7.35) + (7.36) $\Rightarrow u_\varepsilon(x,t) < 0$ on $M \times [0,\tau]$

Claim 2 : $u_\varepsilon(x,t) \leq 0$ for all $x \in M$, $t \in [0,T]$

pf by contradiction

$\frac{\varphi}{\phi} > 0 \Rightarrow \exists$ smallest t_1 and $x_1 \in M$ s.t. $u_\varepsilon = 0$.

$$0 = \nabla u_\varepsilon(x_1,t_1) = \nabla\left(\frac{\varphi}{\phi}\right)(x_1,t_1),$$

$$0 \geq \Delta u_\varepsilon(x_1,t_1) = \Delta\left(\frac{\varphi}{\phi}\right)(x_1,t_1),$$

$$0 \leq \frac{\partial u_\varepsilon}{\partial t}(x_1,t_1) = \frac{\partial}{\partial t}\left(\frac{\varphi}{\phi}\right)(x_1,t_1) - \varepsilon.$$

substitute into ($\star$) $\Rightarrow \varepsilon < 0$ ↯

Claim 1+2 $\Rightarrow$ Cor 7.45.

- $u_\varepsilon \leq 0 \Rightarrow \frac{\varphi}{\phi} \leq 0$ as $\varepsilon \to 0$
- ϕ is the barrier function $\Rightarrow \phi > 0 \Rightarrow \varphi \leq 0$

compact case

Exercise 2.15. *Formulate the minimum principle for supersolutions of the equation*

$$\frac{\partial}{\partial t}u \geq \Delta_{g(t)}u + \langle X(t), \nabla u\rangle + F(u). \tag{2.7}$$

Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold (or any solution where we can apply the maximum principle to the evolution equation for the scalar curvature). Since $|\mathrm{Rc}|^2 \geq \frac{1}{n}R^2$ (see Exercise 1.50), equation (2.2) implies

$$\frac{\partial}{\partial t}R \geq \Delta R + \frac{2}{n}R^2. \tag{2.8}$$

valid

Since the solutions to the ODE $\frac{d\rho}{dt} = \frac{2}{n}\rho^2$ are $\rho(t) = \frac{n}{n\rho(0)^{-1}-2t}$, by the maximum principle one has

$$\boxed{R(x,t) \geq \frac{n}{n\left(\inf_{t=0} R\right)^{-1} - 2t}} \tag{2.9}$$

for all $x \in M^n$ and $t \geq 0$. It $\rho(0) > 0$, then $\rho(t)$ tends to infinity in finite time. When M is closed, let $R_{\min}(0) \doteqdot \inf_{t=0} R$. Hence

Eqn (2.9) holds for noncompact manifolds.

Exercise 7.47. *Show that if we have a complete solution to the Ricci flow with bounded curvature (on compact time intervals) on a noncompact manifold, then inequality (2.9), i.e.,*

$$R(x,t) \geq \frac{n}{n\left(\inf_{t=0} R\right)^{-1} - 2t},$$

still holds.

uniform bound on curvature gives growth control

compute $\frac{d}{dt}\rho = \frac{2}{n}\rho^2 \Rightarrow \rho = \frac{n}{n\rho(0)^{-1}-2t}$

and apply maximum principle to $R-\rho$.

In general, we have the result for a (p,q)-tensor

Lemma 7.48. *Let $(M^n, g(t))$, $t \in [0,T)$, be a complete solution to the Ricci flow with bounded curvature and let α_0 be a (p,q)-tensor with*

$$|\alpha_0(x)|_{g(0)} \leq e^{A(d(x,p)+1)}$$

for some $A < \infty$. Let $E^{p,q} \doteqdot (\bigotimes^p T^*M) \otimes (\bigotimes^q TM)$ and suppose that

$$F_t : E^{p,q} \to E^{p,q}$$

is a fiberwise linear bundle map with

$$\|F_t\|_\infty \doteqdot \sup_{\substack{\beta(x)\in E^{p,q} \\ x\in M}} \frac{|F(\beta(x))|_{g(t)}}{|\beta(x)|_{g(t)}} < \infty.$$

Then there exists $B < \infty$ and a solution $\alpha(t)$, $t \in [0,T)$, of

$$\frac{\partial}{\partial t}\alpha = \Delta_{g(t)}\alpha + F_t(\alpha)$$

with $\alpha(0) = \alpha_0$ and $|\alpha|_{g(t)} \leq e^{B(d(x,p)+1)}$. This solution is unique among all solutions with $|\alpha|_{g(t)} \leq e^{C(d(x,p)+1)}$ for some $C < \infty$.

§ 7.5.

closed + Ricci-DeTurck flow

$\longrightarrow$ short time existence

noncompact + Ricci-DeTurck + Dirichlet BVP.

a family of diffeo (require some work)
pullback the solution of Ricci-DeTurck

Ricci-DeTurck flow

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2Ric_{ij} + \nabla_i W_j + \nabla_j W_i \\ g(0) = g_0 \end{cases}$$

where $W_j = g_{jk} g^{pq}\left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right)$ time depdt.

$\tilde{g}$ auxiliary

Uniqueness Prop 7.51
If $|\tilde{Rm}| \leq A < \infty$, solution to the Ricci-DeTurck flow is unique in the class of solution with

$$\frac{1}{A}\tilde{g} \leq g(t) \leq A\tilde{g}$$

$$|\tilde{\nabla} g(t)| + \sqrt{t}\,|\tilde{\nabla}^2 g(t)| \leq A$$

pf. write RDT in local coordinates
Suppose $g, \tilde{g}$ are two solutions

Check $\left(\frac{\partial}{\partial t} - g^{j\ell} \tilde{\nabla}_j \tilde{\nabla}_\ell\right) |g-\tilde{g}|^2$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - g^{j\ell}\tilde{\nabla}_j\tilde{\nabla}_\ell\right)|g-\bar{g}|^2 &\leq -2g^{pq}\tilde{g}^{ij}\tilde{g}^{j\ell}(\tilde{\nabla}_p g_{ij} - \tilde{\nabla}_p\bar{g}_{ij})(\tilde{\nabla}_q g_{j\ell} - \tilde{\nabla}_q\bar{g}_{j\ell}) \\ &\quad + \frac{C_1A}{\sqrt{t}}|g-\bar{g}|^2 + C_1A|\tilde{\nabla}(g-\bar{g})||g-\bar{g}| \\ &\leq -g^{pq}\tilde{g}^{ij}\tilde{g}^{j\ell}(\tilde{\nabla}_p g_{ij} - \tilde{\nabla}_p\bar{g}_{ij})(\tilde{\nabla}_q g_{j\ell} - \tilde{\nabla}_q\bar{g}_{j\ell}) \\ &\quad + \left(\frac{C_1A}{\sqrt{t}} + \left(\frac{C_1A}{2}\right)^2\right)|g-\bar{g}|^2.\end{aligned}$$

+ maximum principle

Existence

Ω_k precompact exhausion

(D_k)
$$\begin{cases}\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i & \text{in } \Omega_k \times [0,T], \\ g(x,t) = g_0(x) & \text{on } \partial\Omega_k \times [0,T], \\ g(x,0) = g_0(x) & \text{on } \Omega_k \times \{0\}.\end{cases}$$

$\exists!$ $g_k(t)$ sol. on $\Omega_k \times [0, \eta_k]$ (short time $\exists$)

We want to show $\eta_k \not\to \eta \neq 0$ as $k \to \infty$.

Prop. 7.52. (C^0-estimate)

- $g(t)$ solution to (D_k)
- $\widetilde{Rm}(g_0)$ unif. bounded on M

▷ $\forall \delta > 0$, $\exists T = T(\delta, n, \sup_M |\widetilde{Rm}|)$ const.

s.t. $\forall (x,t) \in \Omega_k \times [0,T]$.

$$(1-\delta) g_0(t) \le g(x,t) \le (1+\delta) g_0(t)$$

pf. Consider

$$\varphi(x,t) = \sum_{i=1}^{n} \lambda_i^{-m}(x,t)$$

evalue of g w.r.t. $\tilde{g} = g_0$

long computation gives

$= \sup_M |\widetilde{Rm}|$

$$\frac{\partial \varphi}{\partial t} - g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \varphi \le 2m\varphi \left(\sum_q \lambda_q^{-1}\right) A_0$$

$$\le 2m A_0 n^{1-\frac{1}{m}} \varphi^{1+\frac{1}{m}}$$

maximum principle gives

$\forall \delta > 0$, $\exists T > 0$ s.t. $g(x,t) \ge (1-\delta)\tilde{g}(x)$ on $\Omega_k \times [0,T]$.

The other half uses $\psi = \lambda_i^m$.

Prop 7.53 (global estimate for $|\tilde{\nabla} g|$)

$(M, g(t))$ $\quad t \in [0, T)$

sol. of RDT. on $B_{\tilde{g}}(x_0, \gamma+\delta) \times [0, T)$.

$\exists\, C = C(n, m, \gamma, \delta, T, \tilde{g}) < \infty$. s.t.

if

$$\left(1 - \frac{1}{256000n^{10}}\right)\tilde{g} \le g(t) \le \left(1 + \frac{1}{256000n^{10}}\right)\tilde{g},$$

then $|\tilde{\nabla}^m g|_{\tilde{g}} \le C$ on $B_{\tilde{g}}(x_0, \gamma+\delta) \times [0, T)$.